ÉRETTSÉGI VIZSGA ● 2014. május 6.

MATEMATIKA
ANGOL NYELVEN

EMELT SZINTŰ ÍRÁSBELI
ÉRETTSÉGI VIZSGA

JAVÍTÁSI-ÉRTÉKELÉSI
ÚTMUTATÓ

EMBERI ERŐFORRÁSOK
MINISZTÉRIUMA

Instructions to examiners

Formal requirements:

1. Mark the paper in **ink, different in colour** from the one used by the candidate. Indicate the errors, incomplete solutions, etc in the conventional way.
2. The first one of the rectangles under each problem shows the maximum attainable score on that problem. The **points** given by the examiner are to be entered **in the rectangle** next to that.
3. **If the solution is perfect,** it is enough to enter the maximum scores in the appropriate rectangles.
4. If the solution is incomplete or incorrect, please indicate the individual **partial scores** in the body of the paper, too.
5. Do not assess anything except diagrams that is written **in pencil**.

Assessment of content:

1. The markscheme may contain more than one solution to some of the problems. If the **solution by the candidate is different**, allocate the points by identifying the parts of the solution equivalent to those of the one given in the markscheme.
2. The subtotals in the markscheme can be **further divided**, but the scores awarded should always be integers.
3. If there is a **calculation error** or inaccuracy in the solution, only take off the points for that part where the error occurs. If the reasoning remains correct, with the error carried forward, and the nature of the task does not change, then the points for the rest of the solution should be awarded.
4. In the case of a **principal error,** no points should be awarded at all for that section of the solution, not even for steps that are formally correct. (These logical sections of the solutions are separated by double lines in the markscheme.) However, if the wrong information obtained owing to the principal error is carried forward to the next section or to the next part of the problem and is used correctly there, the maximum score is due for the next part, provided that the error has not changed the nature of the task to be completed.
5. Where the markscheme shows a remark or unit **in brackets**, the solution should be considered complete without that remark or unit as well.
6. If there are **more than one different approaches** to a problem, assess only the one indicated by the candidate.
7. **Do not give extra points** (i.e. more than the score due for the problem or part of problem).
8. A **Do not take off points** for steps or calculations that contain errors but are not actually used by the candidate in the solution of the problem.
9. **Assess only four out of the five problems in Section II.** The candidate was requested to indicate in the appropriate square the number of the problem not to be assessed and counted in their total score. Should there be a solution to that problem, it does not need to be marked. However, if it is still not clear which problem the candidate does not want to be assessed, assume automatically that it is the last one in the question paper, and do not assess that problem.

I

1. a)		
$2x-\frac{\pi}{6}=\frac{\pi}{2}+2k\pi$, where $k\in\mathbf{Z}$.	2 points	
That is, $x=\frac{\pi}{3}+k\pi$ $(k\in\mathbf{Z})$.	2 points	*Award at most 1 point if the period is not divided by 2.*
Checking.	1 point	*This point is for checking the roots within one period by substitution, or referring to equivalent transformations.*
Total:	**5 points**	

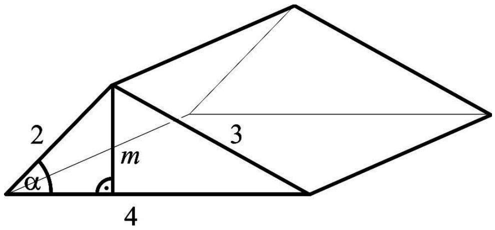
Remarks.
1. 1. Award at most 4 point altogether if the condition $k\in\mathbf{Z}$ *does not appear at all.*
2. Award at most 3 point altogether if the candidate calculates in degrees.
3. Award at most 2 point altogether if no periods appear at all. (1 point for $x=\frac{\pi}{3}$ *and 1 point for checking.)*

1. b)		
The logarithms are only defined if $x>0$.	1 point	*This point is also due if the candidate checks the solution by substitution.*
$\log_9 x=\frac{\log_3 x}{\log_3 9}=$	1 point	
$=\frac{\log_3 x}{2}$	1 point	
Hence from the original equation ($\log_3 x+\frac{\log_3 x}{2}=6$, that is) $\log_3 x=4$.	1 point	
Therefore $x = 81$.	1 point	
Checking.	1 point	*This point is for checking by substitution, or referring to equivalent transformations.*
Total:	**6 points**	

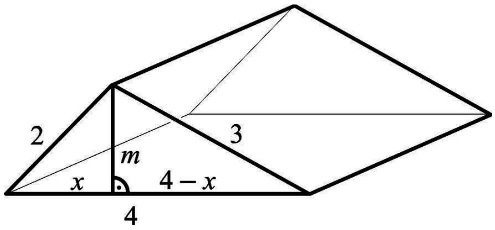
2. a) **Solution 1**		
The number of edges in a complete graph of 16 points is $\binom{16}{2}(=120)$.	1 point	
According to the given information, the number of red edges is $\frac{16\cdot 3}{2}(=24)$.	2 points	*Do not divide.*
(The selection of two points at random is equivalent to the selection of an edge at random.) Thus the probability of the selected edge being red is $p=\frac{24}{120}\left(=\frac{1}{5}=0.2\right)$.	1 point	
Total:	**4 point**	

2. a) **Solution 2**		
(Since the points of the graph are equivalent,) consider either one of the two points selected.	1 point	
There are 15 edges starting at this point,	1 point	
3 of which are red.	1 point	
The probability of the edge from this point to the other point selected being red is $p=\frac{3}{15}\left(=\frac{1}{5}=0.2\right)$.	1 point	
Total:	**4 points**	

2. b)		
The number of edges in a tree of n points is $n-1$.	1 point	*These 2 points are also due if these relationships are correctly applied in accordance with the conditions of the problem.*
The number of edges in a complete graph on n points is $\frac{n(n-1)}{2}$.	1 point	
According to the given information, $\frac{n(n-1)}{2}-45=n-1$,	1 point	
hence $n^2-3n-88=0$.	1 point	
Solved: $n=11$ or $n=-8$.	1 point	
(Since n is a positive integer,) $n=-8$ is not a solution of the problem.	1 point	
The graph has 11 points.	1 point	
Checking: The complete graph on 11 points has 55 edges, a tree of 11 points has 10 edges, and 55 – 45 = 10. Therefore this is a solution.	1 point	
Total:	**8 points**	

3.		
		
The bronze model fits in the box if its height (*m*) does not exceed 1.5 cm.	1 point	*The point is also due if this idea is only reflected by the solution.*
Applying the cosine rule to the angle *α* of the triangular face: $3^2 = 2^2 + 4^2 - 2\cdot 2\cdot 4\cdot \cos\alpha$.	1 point*	
Hence $\cos\alpha = \frac{11}{16} (= 0.6875)$,	1 point*	
$\alpha \approx 46.57°$.	1 point*	
Calculation of the height *m* drawn to the 4-cm side of the triangular face: $\sin 46.57° = \frac{m}{2}$.	1 point*	
Hence $m \approx 1.45$ (cm).	1 point*	
1.45 cm < 1.5 cm, so the bronze model fits in the box.	1 point	
If the object is turned so that it stands on a triangular face, it makes a triangular right prism of height $M = 4$ (cm),	2 points	*The 2 points are also due if this idea is only reflected by the solution.*
and volume $V = T \cdot M \approx \frac{4\cdot 1.45}{2}\cdot 4 =$	1 point	
$= 11.6\ \text{cm}^3 =$	1 point	
$= 0.0116\ \text{dm}^3$.	1 point	
The mass of the bronze gift is $0.0116\cdot 8.2 = 0.09512$ (kg).	1 point	
This is about 95 g, therefore it does not exceed 100 g.	1 point	
Total:	**14 points**	

*Remark. The 5 points marked with * are also due for either of the reasonings below:*

		
(The height m divides the triangular face into two right-angled triangles. Applying the Pythagorean theorem to these.) $m^2 + x^2 = 2^2$,	1 point	
$m^2 + (4 - x)^2 = 3^2$.	1 point	
From the second equation, by expanding and rearranging: $m^2 = -7 + 8x - x^2$.	1 point	
Substituted in the first equation and rearranged: $x = \frac{11}{8}$.	1 point	
Substituted back: $m \left(= \frac{3\sqrt{15}}{8} \right) \approx 1.45$ (cm).	1 point	

(The area of the triangle is obtained from the three sides of the triangle by applying Hero's formula.) The semi-perimeter is $s = 4.5$.	1 point	
So $T = \sqrt{4.5 \cdot 0.5 \cdot 1.5 \cdot 2.5} \approx$	1 point	
≈ 2.9 (cm^2).	1 point	
The height of the triangle is obtained from this by using another area formula: $2.9 \approx \frac{4m}{2}$.	1 point	
$m \approx 1.45$ (cm)	1 point	

4. a)		
If x denotes the missing seventh item then the mean of the seven data is $\frac{25+x}{7}$.	1 point	
The value of the mode is 2 for any value of x.	1 point	
Depending on the unknown item, the median may have three different values. I. If x is less than 3 (1 or 2) then the median is 2.	1 point	
This is impossible since the mode is also 2, and thus the sequence would not be strictly increasing.	1 point	
II. If the value of x is 3, then the median is also 3 and the mean is 4.	1 point	
Since the numbers 2, 3, 4 form an (increasing) arithmetic sequence, $x = 3$ is a solution of the problem.	1 point	
III. If x is greater than 3, the value of the median is 4 and the mean is more than 4.	1 point	
Then the three terms in increasing order are 2; 4; $\frac{25+x}{7}$.	1 point	
These form an arithmetic sequence if $\frac{25+x}{7} = 6$, that is, $x = 17$.	1 point	
Total:	**9 points**	

4. b)		
Since the numbers in question are even, they may only end in 0, 2 or 4.	1 point	*The point is also due if this idea is only reflected by the solution.*
If the number ends in 0, there are 5, 4 and 5 choices, respectively in the places of thousands, hundreds and tens. Thus there are $5 \cdot 4 \cdot 3 (= 60)$ such numbers.	1 point	
If the number ends in 2, or 4, there are 4, 4 and 3 choices, respectively in the places of thousands, hundreds and tens (since a number cannot begin with a 0). Thus there are $2 \cdot 4 \cdot 4 \cdot 3 (= 96)$ such numbers.	2 points	
Thus there are 156 four-digit numbers altogether that meet the conditions of the problem.	1 point	
Total:	**5 points**	

II

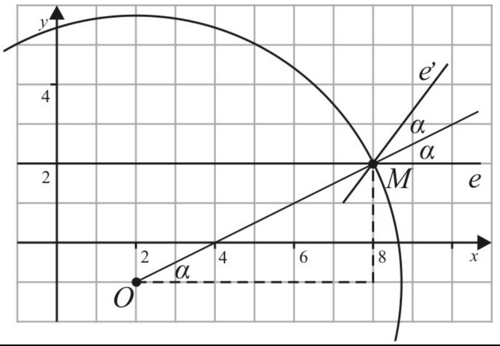
5. a)		
If w denotes the number of female employees in the department and m denotes the number of males, it follows from the given information that the total age of the women is $40w$ and the total age of the men is $44m$.	1 point	*The point is also due if this idea is only reflected by the solution.*
It also follows that $\frac{40w+44m}{w+m}=41.5$.	2 points	
Hence $m = 0.6w$.	2 points	
That is, there are 0.6 times as many male employees as female employees in the department.	1 point	
Total:	**6 points**	

5. b)		
It is given that after the reorganization, $\frac{m-7}{w-9}=\frac{1}{2}$.	1 point	
For the initial numbers, $w = 1.5m$. Substituted in the equation above, $\frac{m-7}{1.5m-9}=\frac{1}{2}$.	1 point	
Hence $m = 10$.	1 point	
Substituted back, $w = 15$ is obtained.	1 point	
After the reorganization, (15 – 9 =) 6 women and (10 – 7 =) 3 men remained in the department.	1 point	
Total:	**5 points**	

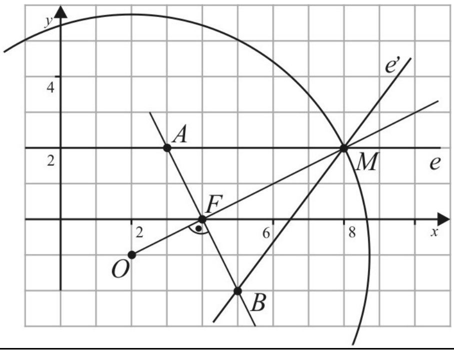
5. c) Solution 1		
The number of ways to select the members of the first team: $\binom{6}{2}\cdot\binom{3}{1}(=45)$.	1 point	
The number of ways to select the second team: $\binom{4}{2}\cdot\binom{2}{1}(=12)$.	1 point	
The third team is determined by the selection of the first two teams.	1 point	
The number of all cases is the product of the above results (if the order of the teams is taken in consideration): $\binom{6}{2}\cdot\binom{3}{1}\cdot\binom{4}{2}\cdot\binom{2}{1}(=540)$.	1 point	
Since the order of the teams is disregarded, the number of possibilities is $\frac{540}{3!}=90$.	1 point	
Total:	**5 points**	

5. c) Solution 2		
Let the three men be *A*, *B* and *C*.	1 point	*The point is also due if this idea is only reflected by the solution.*
The number of ways to select two women for the team of *A* is $\binom{6}{2}(=15)$.	1 point	
The number of ways to select two out of the four remaining women for the team of *B* is $\binom{4}{2}(=6)$.	1 point	
The two women still remaining will be in the team of *C*.	1 point	
Thus there are $\binom{6}{2}\cdot\binom{4}{2}=90$ ways altogether to form the teams of three.	1 point	
Total:	**5 points**	

5. c) Solution 3		
The number of possible ways to divide the women into three pairs is $\frac{6!}{(2!)^3}(=90)$ (if the order of the pairs matters).	2 points	
Since the order of the teams is not important, the number of possibilities is $\frac{90}{3!}(=15)$.	1 point	
Given the three pairs formed by the women, there are $3!(=6)$ ways for the three men to join them.	1 point	
Thus the number of all cases is the product of the results above: $15\cdot 3!=90$.	1 point	
Total:	**5 points**	

6. a) **Solution 1**		
Correct diagram (showing the centre of the circle, the line *e* and the reflection correctly). 	1 point	*This point is also due if there is no diagram but the calculation is correct.*
The centre of the circle is $O(2; -1)$.	1 point	
By substituting $y = 2$ in the equation of the circle, the equation $(x-2)^2 = 36$ is obtained.	1 point	
The positive solution of the equation is $x = 8$, so the coordinates of point M are $(8; 2)$.	1 point	
A direction vector of line OM is $(6; 3)$,	1 point	
thus its slope (the tangent of the angle from the positive x-axis) is $\frac{3}{6}\left(=\frac{1}{2}\right)$.	1 point	
If line OM encloses an angle α with the x-axis then it follows that the angle of its reflection with the x-axis is 2α.	1 point	*The point is also due if this idea is only reflected by the solution.*
Therefore the slope of the reflection is $\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} =$	1 point	$\alpha \approx 26.565°$ $2\alpha \approx 53.13°$
$= \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$.	1 point	$\tan 2\alpha \approx 1.333$
Substituting in the line equation $y = mx + b$ the coordinates of point M and the slope of the reflection, ($2 = \frac{4}{3}\cdot 8 + b$) we get $b = -\frac{26}{3}$.	2 points	$b \approx -8.664$
The equation of the reflection is $y = \frac{4}{3}x - \frac{26}{3}$.	1 point	$y = 1.333x - 8.664$
Total:	**12 points**	

Remark. Award full mark if the candidate uses approximate values and correct rounding to express the equation of the reflection correctly.

6. a) **Solution 2**		
Correct diagram (showing the centre of the circle, the line e and the reflection correctly).  	1 point	*This point is also due if there is no diagram but the calculation is correct.*
The centre of the circle is $O(2; -1)$.	1 point	
By substituting $y = 2$ in the equation of the circle, the equation $(x-2)^2 = 36$ is obtained.	1 point	
The positive solution is $x = 8$, so $M(8; 2)$.	1 point	
A direction vector of line OM is (6; 3),	1 point	
so its equation is $x - 2y = 4$.	1 point	
Let B denote the reflection of an arbitrary point A of line e in the line OM. Thus the reflection of line e in the line OM will be the line BM.	1 point	*The point is also due if this idea is only reflected by the solution.*
Therefore we can select any point of the line $y = 2$, for example, the point with coordinates $A(3; 2)$. The equation of the perpendicular dropped from point A to line OM is $2x + y = 8$.	1 point	
The intersection F of this line with line OM is calculated by solving the simultaneous equations $\left.\begin{array}{l} x-2y=4 \\ 2x+y=8 \end{array}\right\}$.	1 point	
The solution of these is $x = 4$ and $y = 0$, so $F(4; 0)$.	1 point	
The reflection of point A in F is $B(5; -2)$.	1 point	
The equation of the line passing through the points M and B is $-4x + 3y = -26$.	1 point	
Total:	**12 points**	

6. b)		
By substituting $y = 2$ in the equation of the parabola, we get that the intersection of the line and parabola in the first quadrant is the point $P(3; 2)$.	1 point	
The line in question is the tangent drawn to the parabola at point $P(3; 2)$. Its slope is the value of the derivative function of the function $f: \mathbb{R}^+ \to \mathbb{R},\ f(x) = -x^2 + 2x + 5$ at the point $x = 3$.	1 point	*The point is also due if this idea is only reflected by the solution.*
$f'(x) = -2x + 2$.	1 point	
Thus the slope in question is $m = f'(3) = -4$.	1 point	
Total:	**4 points**	

7. a)		
(Substituting $x = 1$ and $x = -1$) $1 + a + b + c = -1 + a - b + c + 4$,	2 points	
therefore $b = 1$.	1 point	
$f'(x) = 3x^2 + 2ax + 1$,	2 points	
$f'(3) = 27 + 6a + 1 = 10$.	1 point	
Hence $a = -3$.	1 point	
$\int_0^2 f(x)dx = \int_0^2 (x^3 - 3x^2 + x + c)\, dx =$ $= \left[\frac{x^4}{4} - \frac{3x^3}{3} + \frac{x^2}{2} + cx\right]_0^2 =$	2 points	
$= 4 - 8 + 2 + 2c$.	1 point	
(According to the conditions,) $4 - 8 + 2 + 2c = -8$, therefore $c = -3$.	1 point	
Total:	**11 points**	

7. b)		
$x^3 - 3x^2 + x - 3 = x^2 \cdot (x-3) + x - 3 =$	1 point	
$= (x-3) \cdot (x^2 + 1)$	1 point	
$g(x) = (x-3) \cdot (x^2 + 1) = 0$	1 point	
(The value of a product is zero exactly if one of the factors is zero,) so $x = 3$.	1 point	
Since $x^2 + 1 > 0$ for all real x, the function has no other zeros.	1 point	
Total:	**5 points**	

8. a) **Solution 1**		
There are $\binom{32}{2}$ (= 496) ways altogether to select two cards.	1 point	
The number of pairs of cards of different suits (i.e. the number of favourable cases) is $\binom{4}{2}\cdot\binom{8}{1}\cdot\binom{8}{1}$ (= 384)	2 points	
The probability in question: $p=\frac{\binom{4}{2}\cdot\binom{8}{1}\cdot\binom{8}{1}}{\binom{32}{2}}=\frac{384}{496}\left(=\frac{24}{31}\right)\approx 0.774.$	1 point	
Total:	**4 points**	

8. a) **Solution 2**		
Suppose the dealing starts by placing two cards in the talon.	1 point	*The point is also due if this idea is only reflected by the solution.*
The first card may me any card.	1 point	
There are 31 ways to get the second card, and 24 out of the 31 cards belong to a different suit.	1 point	
The probability in question: $p=\frac{24}{31}\approx 0.774.$	1 point	
Total:	**4 points**	

8. b)		
There are $\binom{32}{10}$ ways altogether for Elemér to get 10 cards.	1 point	
(Since there are four suits, so two cards need to be selected from the remaining 24 cards,) the number of favourable cases is $4\cdot\binom{8}{8}\cdot\binom{24}{2}=4\cdot\binom{24}{2}$.	2 points	*Award 1 point for the partial result* $\binom{8}{8}\cdot\binom{24}{2}$.
The probability in question: $p=\frac{4\cdot\binom{24}{2}}{\binom{32}{10}}=\frac{1104}{64\,512\,240}\approx 0.000017.$	1 point	
Total:	**4 points**	

8. c)		
(Let A denote the event that Fanni has at least one ace. The probability of event A is calculated from its complement:) $P(A) = 1 - P(\overline{A}) =$	1 point	
$= 1 - \dfrac{\binom{28}{10}}{\binom{32}{10}} =$	1 point	
$= 1 - \dfrac{13\,123\,110}{64\,512\,240} \approx 0.7966$ is the probability that there is at least one ace in Fanni's hand.	1 point	
Total:	**3 points**	

8. d)		
(Let B denote the event that all four aces are with Fanni, and let A be the event that she has at least one.) By definition of conditional probability, $P(B\|A) = \dfrac{P(BA)}{P(A)}$.	1 point	*The point is also due if this idea is only reflected by the solution.*
(if all aces are with Fanni then she has at least one, so) $P(BA) = P(B)$.	1 point	
There are $\binom{4}{4}\binom{28}{6}$ cases when the four aces are all in Fanni's hand,	1 point	
therefore $P(B) = \dfrac{\binom{4}{4}\binom{28}{6}}{\binom{32}{10}} = \dfrac{376\,740}{64\,512\,240} \approx 0.00584$.	1 point	
So the probability in question: $P(B\|A) \approx \dfrac{0.00584}{0.7966} \approx 0.00733$.	1 point	
Total:	**5 points**	

Remarks:
1. Accept correct answers in any form (common fraction, decimal or percentage).
2. Other answers obtained by reasonable and correct rounding may also be accepted.

9.		
If Rita's score increased by a factor of k in each round and Péter's score increased by a factor of n ($n > k > 1$), and the initial score is x, then Rita's scores in the successive rounds are $x; kx; k^2x; k^3x$, and Péter's scores are $x; nx; n^2x; n^3x$.	2 points	
The differences between their scores in the successive rounds are $nx - kx = 20$; $n^2x - k^2x = 70$; $n^3x - k^3x = 185$.	2 points	
$x(n - k) = 20$ and $x(n^2 - k^2) = x(n-k)(n+k) =$	1 point	
$= 20(n+k) = 70$.	1 point	
Hence $n + k = 3.5$, therefore $k = 3.5 - n$.	1 point	
$x(n^3 - k^3) = x(n-k)(n^2 + nk + k^2) = 185$	1 point	
Substituting the results above in this equation, $20 \cdot \left[(n^2 + n(3.5 - n) + (3.5 - n)^2\right] = 185$.	1 point	
Rearranged: $2n^2 - 7n + 6 = 0$.	2 points	
The roots are $n_1 = 2$ and $n_2 = 1.5$.	1 point	
(Since $n + k = 3.5$,) $k_1 = 1.5$ and $k_2 = 2$.	1 point	
(Since $n > k$,) only $n = 2$ and $k = 1.5$ give a solution.	1 point	
The initial score was $\left(x = \frac{20}{n-k} = \frac{20}{0.5} = \right)40$.	1 point	
Checking. Rita's scores in succession are 40, 60, 90 and 135; Péter's scores are 40, 80, 160 and 320; and the differences in the successive rounds are 0, 20, 70 and 185.	1 point	
Total:	**16 points**	